The Methods for Efficient Transverse Vibration of Structural Dynamics: a Historical Point of View

A. Rugerri Toni Liong
Mechanical Engineering Department
Faculty of Engineering, Atma Jaya Catholic University of Indonesia
Jalan Jenderal Sudirman 51, Jakarta 12930, Indonesia
E-mail: rugerri.liong@atmajaya.ac.id

ABSTRACT

The different assumptions and corresponding theories of transverse vibrations of beams are presented. Eight mathematical models, namely Euler-Bernoulli, Rayleigh, Euler-Bernoulli modified, Bress, Volterra, Ambartsumyan, Timoshenko and Love theory are firstly discussed. Then, different theories of dynamic behaviours of beams can be obtained from the equations of the theory of elasticity, which are presented with respect to average values. Nowadays, beam elements are usually based on two theories, i.e Euler-Bernoulli and Timoshenko beam theory. One of the recent developments pointed out is the computer program ABAQUS, a finite element based cross-sectional and non-linear analysis for beam. Finally, several examples are presented as numerical validations in which FE results are compared with those from the theories in the literature where available and appropriate.

***Keywords**: Transverse vibration of beams, analytical model, finite element.*

INTRODUCTION

The equations describing the mechanic of a three dimensional continuum are formidable to solve even for a simple constitutive model like isotropic hyper elasticity. In the age of computer and the finite element method, it is still not feasible to treat every solid body as a three-dimensional continuum. Bodies with certain geometric features are amenable to a reduction from three dimensions to fewer dimensions, from the perspective of the governing differential equations [1]. These bodies are usually called beams (one dimension), plates (two dimensions, flat), and shells (two dimensions, curved). These reduced theories comprise of a subset of solid mechanics generally referred to as structural mechanics.

This paper really has two complementary purposes. First, it provides a careful and thorough derivation of the equations of linear beam theory in the context of the three-dimensional solid mechanics. The great merit of approaching the derivation the relevant application of the general theory is that can be seen where is the strengths and limitations of beam theory. Second, the ordinary differential equations of beam theory are much more likely to yield classical solutions than are the partial differential equations of the three-dimensional theory. Because classical and variational approaches are both relatively easy, beam theory provides a fertile opportunity for the study of the relationship between classical and variational methods [1].

A great number of practical problems can be idealized as planar problems. The assumption of planar behaviour comes at a fairly high price, the cost of which we can clearly see from the equations for the beam in three dimensions. First, the loading must be such that it does not excite out-of-plane motions. Second, the cross sections must be symmetric with respect to the plane of loading. Clearly, the centroid of the section will lie on the line of symmetry, and this line should be taken as the coordinate axis.

Unfortunately, beam theory is not completely consistent with the three-dimensional theory. Every time a system is constrained, a price must be paid. In the present case, the deformation map so that cross sections of the beam remain rigid will be constrained. It must be paid for this simplification in the constitutive equations and in the satisfaction of equilibrium locally within a cross section [1-3].

TRANSVERSE VIBRATION OF BEAMS

Euler-Bernoulli theory (1735-1744)

The transverse or lateral vibration of a thin uniform beam is another vibration problem in which both elasticity and mass are distributed. Consider the moments and forces acting on the element of the beam shown in Figure 1. The beam has a cross-sectional area A, flexural rigidity EI, density of material ρ.

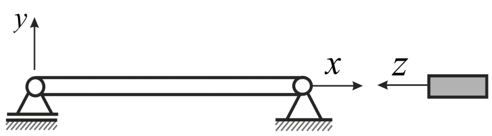


Figure 1. Rectangular beam and its cross section

$$\lambda^4 = \frac{\rho A \omega^2}{EI} \tag{1}$$

and the constants $C_{1,2,3,4}$ are determined from the boundary conditions.

For simplicity, it is assumed that damping is very small, therefore it can be neglected and for synchronous whirl $\dot{\theta} = \Omega$, we get

$$\omega = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}} \qquad n = 1, 2, \cdots \tag{2}$$

This elementary beam theory is a most simplest and valid only when the height of the beam is small compared with its length [8].

Rayleigh theory (1877)

The Rayleigh theory takes into account the effect of rotary inertia and neglects the effect of shear deflection. This theory has another assumptions that the cross-sections remain plane and orthogonal to the neutral axis and the longitudinal fibres do not compress each other.

Differential equation of transverse vibration of the Rayleigh beam

$$\frac{\partial^4 y}{\partial x^4} + \frac{\rho A}{EI}\frac{\partial^2 y}{\partial t^2} - \frac{\rho}{E}\frac{\partial^4 y}{\partial x^2 \partial t^2} = 0 \tag{3}$$

Using the general solution of partial differential equations, we get the natural frequency

The Euler-Bernoulli theory takes into account the inertia forces due to the transverse translation and neglects the effect of shear deflection and rotary inertia. This theory has another assumption that the cross-sections remain plane and orthogonal to the neutral axis and the longitudinal fibres do not compress each other.

The general equation for the transverse vibration of a uniform beam is [4-7]

$$\frac{\partial^4 y}{\partial x^4} + \frac{\rho A}{EI}\frac{\partial^2 y}{\partial t^2} = 0 \tag{4}$$

When a beam performs a normal mode of vibration the deflection at any point of the beam varies harmonically with time, and can be written

$$\begin{aligned} y &= C_1 \cos\lambda x + C_2 \sin\lambda x \\ &\quad + C_3 \cosh\lambda x + C_4 \sinh\lambda x \end{aligned} \tag{5}$$

where

$$\omega = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}} \cdot \frac{1}{\sqrt{1 + \left(\frac{n\pi}{L}\right)^2 \frac{I}{A}}} \tag{6}$$

The Euler-Bernoulli theory gives good results only for low frequencies. The Rayleigh theory gives a worse result than the modified Euler-Bernoulli theory.

Bress theory (1859)

The Bress theory [8] takes into account the rotational inertia, shear deformation and their combined effect. This theory has assumptions that the cross-sections remain plane and the longitudinal fibres do not compress each other. Differential equation of transverse vibration is written as

$$\frac{\partial^4 y}{\partial x^4} + \frac{\rho A}{EI}\frac{\partial^2 y}{\partial t^2} - \left(\frac{\rho}{E} + \frac{\rho}{G}\right)\frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{\rho^2}{EG}\frac{\partial^4 y}{\partial t^4} = 0 \tag{7}$$

where

$$\omega = \left[\frac{EG}{2\rho}\left\{\frac{A}{EI} + \left(\frac{n\pi}{L}\right)^2\left(\frac{1}{E} + \frac{1}{G}\right) \pm \sqrt{\left(\frac{A}{EI} + \left(\frac{n\pi}{L}\right)^2\left(\frac{1}{E} + \frac{1}{G}\right)\right)^2 - 4\left(\frac{n\pi}{L}\right)^4 \frac{1}{EG}}\right\}\right]^{1/2} \tag{8}$$

In this theory, the third and fourth terms reflect the rotational inertia and the shear deformation, respectively. The last term describes their combined effect, which leads to the occurrence of a cut-off frequency of the model.

Volterra theory (1955)

The Volterra theory [8], as with the Bress theory, takes into account the rotational inertia, shear deformation and their combined effect. This theory has assumptions that all displacements are linear function of the transverse coordinates. Differential equation of transverse vibration is written as

$$\frac{\partial^4 y}{\partial x^4}+\frac{\rho A\left(1-\nu^2\right)}{EI}\frac{\partial^2 y}{\partial t^2}-\left(\frac{\rho\left(1-\nu^2\right)}{E}+\frac{\rho}{G}\right)\frac{\partial^4 y}{\partial x^2\partial t^2}+\frac{\rho^2\left(1-\nu^2\right)}{EG}\frac{\partial^4 y}{\partial t^4}=0 \qquad (9)$$

where

$$\omega=\left[\frac{EG}{2\rho\left(1-\nu^2\right)}\left\{\frac{A\left(1-\nu^2\right)}{EI}+\left(\frac{n\pi}{L}\right)^2\left(\frac{\left(1-\nu^2\right)}{E}+\frac{1}{G}\right)\pm\sqrt{\left(\frac{A\left(1-\nu^2\right)}{EI}+\left(\frac{n\pi}{L}\right)^2\left(\frac{\left(1-\nu^2\right)}{E}+\frac{1}{G}\right)\right)^2-4\left(\frac{n\pi}{L}\right)^4\frac{\left(1-\nu^2\right)}{EG}}\right\}\right]^{1/2} \qquad (10)$$

Difference between Volterra and Bress theories can be seen from their equations, that the bending stiffness of the beam according to the Volterra model is $(1-\nu^2)^{-1}$ times greater than that given by the Bress theory. This is because transverse compressive and tensile stresses are not allowed in the Volterra model.

Ambartsumyan theory (1956)

The Ambartsumyan theory [8] allows the distortion of the cross-section. This theory has assumptions that transverse displacements for all points in the cross-section are equal and the shear stress is distributed according to function $f(y)$. Differential equation of transverse vibration is written as

$$\frac{\partial^4 y}{\partial x^4}+\frac{\rho A\left(1-\nu^2\right)}{EI}\frac{\partial^2 y}{\partial t^2}-\left(\frac{\rho\left(1-\nu^2\right)}{E}+\frac{\rho}{aG}\right)\frac{\partial^4 y}{\partial x^2\partial t^2}+\frac{\rho^2\left(1-\nu^2\right)}{aEG}\frac{\partial^4 y}{\partial t^4}=0 \qquad (11)$$

where

$$\omega=\left[\frac{EG}{2\rho\left(1-\nu^2\right)}\left\{\frac{A\left(1-\nu^2\right)}{EI}+\left(\frac{n\pi}{L}\right)^2\left(\frac{\left(1-\nu^2\right)}{E}+\frac{1}{aG}\right)\pm\sqrt{\left(\frac{A\left(1-\nu^2\right)}{EI}+\left(\frac{n\pi}{L}\right)^2\left(\frac{\left(1-\nu^2\right)}{E}+\frac{1}{aG}\right)\right)^2-4\left(\frac{n\pi}{L}\right)^4\frac{\left(1-\nu^2\right)}{aEG}}\right\}\right]^{1/2} \qquad (12)$$

Difference between Ambartsumyan and Volterra theories can be seen from their equations, that the Ambartsumyan's differential equation differs from the Volterra equation by coefficient a at c_t^2. This coefficient depends on $f(y)$. There is special case that Ambartsumyan and Volterra differential equations coincide if $f(y)=0.5$.

Timoshenko theory (1921)

When a beam is subjected to lateral vibration so that the depth of the beam is a significant proportion of the distance between two adjacent nodes, rotary inertia of beam elements and transverse shear deformation arising from the severe contortions of the beam during vibration make significant contributions to the lateral deflection. Therefore rotary inertia and shear effects must be taken into account in the analysis of high-frequency vibration of all beams, and in all analyses of deep beams [9].

The Timoshenko theory takes into account the rotational inertia, shear deformation and their combined effects. The fundamental assumption for the Timoshenko theory is arbitrary shear coefficient κ enters into the equation. Differential equation of this theory is

$$\frac{\partial^4 y}{\partial x^4}+\frac{\rho A}{EI}\frac{\partial^2 y}{\partial t^2}-\left(\frac{\rho}{E}+\frac{\rho}{\kappa G}\right)\frac{\partial^4 y}{\partial x^2\partial t^2}+\frac{\rho^2}{\kappa EG}\frac{\partial^4 y}{\partial t^4}=0 \qquad (13)$$

For simplicity, it is assumed that damping is very small, therefore it can be neglected and for synchronous whirl $\dot{\theta}=\Omega$, we get

$$\omega=\left[\frac{\kappa EG}{2\rho}\left\{\frac{A}{EI}+\left(\frac{n\pi}{L}\right)^2\left(\frac{1}{E}+\frac{1}{\kappa G}\right)\pm\sqrt{\left(\frac{A}{EI}+\left(\frac{n\pi}{L}\right)^2\left(\frac{1}{E}+\frac{1}{\kappa G}\right)\right)^2-4\left(\frac{n\pi}{L}\right)^4\frac{1}{\kappa EG}}\right\}\right]^{1/2} \qquad (14)$$

The fundamental difference between the Rayleigh and Bress theories, on one hand, and the Timoshenko theory, on the other, is that the correction factor in the Rayleigh and Bress theories appears as a result of shear and rotary effects, whereas in the Timoshenko theory, the correction factor is introduced in the initial equations. The Timoshenko theory gives very high precision results for high frequencies, high modes of a thin and short beam. In this case, the Timoshenko theory gives a better result than the Euler-Bernoulli theory.

Love theory (1927)

The equation of the Love theory [8] may be obtained from the Timoshenko equation as a special case.

$$\frac{\partial^4 y}{\partial x^4}+\frac{\rho A}{EI}\frac{\partial^2 y}{\partial t^2}-\left(\frac{\rho}{E}+\frac{\rho}{\kappa G}\right)\frac{\partial^4 y}{\partial x^2 \partial t^2}=0 \tag{15}$$

where

$$\omega=\left(\frac{n\pi}{L}\right)^2\sqrt{\frac{EI}{\rho A}}\cdot\frac{1}{\sqrt{1+\left(\frac{n\pi}{L}\right)^2\frac{I}{A}\left(1+\frac{2(1+\nu)}{\kappa}\right)}} \tag{16}$$

FINITE ELEMENT METHOD

Many structures are too complicated to be analysed by classical techniques, so that an approximate method has to be used. It can be seen from the receptance analysis of complicated substructures is a useful analytical technique, provided that sufficient computational facilities are available to solve the resulting equations. The finite element method of analysis extends this method to the consideration of continuous structures as a number element, connected to each other by conditions of compatibility and equilibrium. Complicated structures thus be modelled as the aggregate of simpler structures.

The principal advantage of the finite element methods is its generality, it can be used to calculate the natural frequencies and mode shapes of any linear elastic system. However, it is a numerical technique that requires a fairly large computer, and care has to be taken over the sensitivity of the computer output to small changes in input. For beam type systems the finite element method is similar to the lumped mass method. The infinite number of degrees of freedom associated with a continuous system can thereby be reduced to a finite number of degrees of freedom, which can be examined individually.

FORMULATION FOR CASE 1

As a first case, we will investigate the natural frequencies of an elastic simply supported beam as shown in Figure 2. By using the following material and geometrical data: Young's modulus E =210 GPa, Poisson's ratio ν=0.3, κ=5/6, mass density ρ=7850 kg/m^3, length of the beam L=1m, width and height of beam b=0.04 m and h= 0,02 m, respectively.

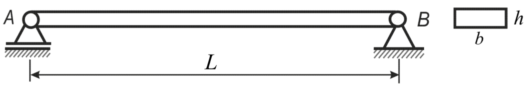


Figure 2. An elastic simply supported beam

Figure 3 shows the natural frequencies of beam until 12th mode numbers with associated their natural frequencies for eight beam theories and the finite element results using software ABAQUS. It can be seen that the natural frequencies until 6th mode shapes are very near between all the theories for case the elastic long beam. The difference of natural frequencies appears upper 6th mode shapes. Figure 4 depicts the natural frequencies only for four theories, which are used commonly nowadays and the difference between these theories is shown in Figure 5.

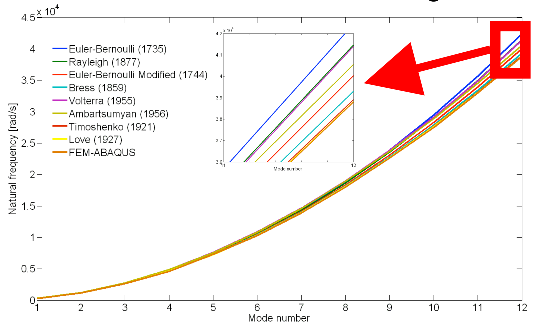


Figure 3. Natural frequencies corresponding to their mode shapes for each beam theories

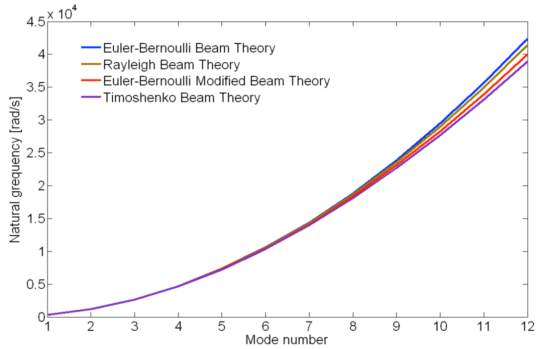


Figure 4. Natural frequencies corresponding to their mode shapes for commonly used beam theories

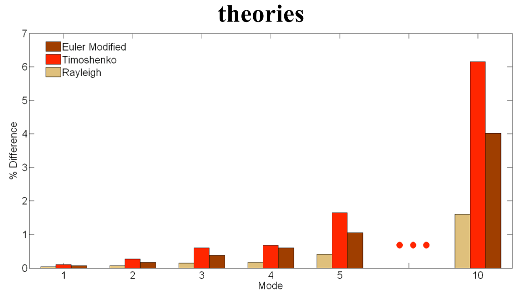
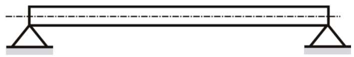


Figure 5. Difference between Euler modified, Timoshenko and Rayleigh theories.

FORMULATION FOR CASE 2

As a second case, the natural frequencies of elastic simply supported beam and cantilever beam using the simplest theory, i.e. Euler-Bernoulli theory and compare its result to the finite element results by using the same of material and geometrical data as case 1.



Mode	Euler-Bernoulli	FEM
1st	294.7	293.6
2nd	1178.9	1174.1
3rd	2652.5	2641.2
4th	4715.6	4692.6
5th	7368.1	7351.2
6th	10610.1	10479.0

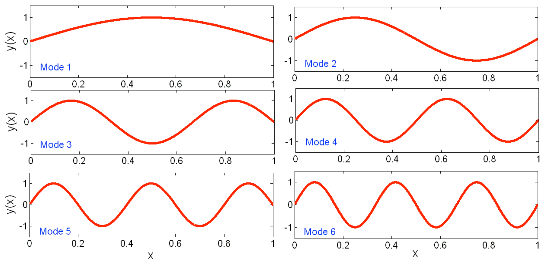


Figure 6. Natural frequencies and mode shapes of elastic simply supported beam.

Mode	Euler-Bernoulli	FEM
1st	105.0	104.4
2nd	658.0	652.6
3rd	1842.4	1832.1
4th	3610.4	3608.1
5th	5968.2	5856.8
6th	8915.4	8878.3

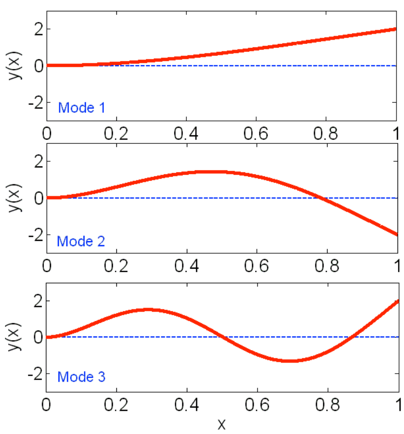


Figure 7. Natural frequencies and mode shapes of elastic cantilever beam.

Figure 6 and 7 show the natural frequencies of elastic simply supported beam and cantilever beam calculated by the Euler-Bernoulli theory and finite element. Difference occurs due to stiffening effect in finite element which finite element model is stiffer than the real structure. In general, displacement results are smaller in magnitudes than the exact values. Furthermore, finite element solution depends strongly on the number of element as shown in Figure 8. Hence, finite element solution of natural frequency and mode shapes provide a lower bound of the exact solution.

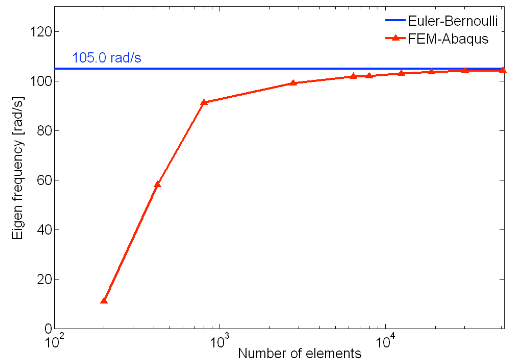


Figure 8. Natural frequencies and mode shapes of elastic simply supported beam.

FORMULATION FOR CASE 3

Nowadays, beam elements are usually based on two theories, i.e. Euler-Bernoulli beam theory and Timoshenko beam theory. In Euler-Bernoulli beams, transverse shear stress is not taken into account whereas in Timoshenko beams transverse shear stresses are taken into account. The reason why transverse shear stress is not taken into account in Euler-Bernoulli beams is bending is assumed to behave in such a way that cross section normal to the neutral axis remain normal to the neutral axis after bending. In case of Timoshenko beams initially cross section in normal to the neutral axis but does not remain normal after bending. Actually Euler-Bernoulli Beam elements give good results for normal stress, because they are capable of capturing bending dominated deformation fields. If a beam is not slender and it goes into bending dominated deformation then Timoshenko elements are weak to capture normal stress and classical beam elements are weak to capture shear deformation.

As the last case, we will investigate the Euler-Bernoulli and the Timoshenko beam theories of an elastic cantilever beam as shown in Figure 9. By using the following material and geometrical data: Young's modulus E =210 GPa, Poisson's ratio ν=0.3, κ=5/6, mass density ρ=7850 kg/m^3, width and height of beam b=0.01 m and h= 0,10 m, respectively. Length of the beam varied from L/h =10, 5, and 2.5.

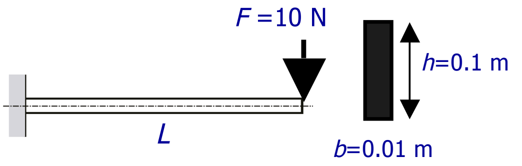


Figure 9. Elastic cantilever beam with varying *L/h*.

The percent of the difference of natural frequency between Euler-Bernoulli beams and Timoshenko beams was obtained for

- L/h = 10 δ = 0.00005 %
- L/h = 5 δ = 0.00042 %
- L/h = 2.5 δ = 0.00340 %

Figure 10 shows the comparison of mode shapes between Euler-Bernoulli beams and Timoshenko beams for variation of L/h of beams. It can be seen that smaller L/h the difference of natural frequencies bigger and the difference of deflection bigger too. δ_b is the deflection only due to bending (Euler-Bernoully theory) and δ_s is the deflection due to shear. Timoshenko theory use $\delta_b + \delta_s$.

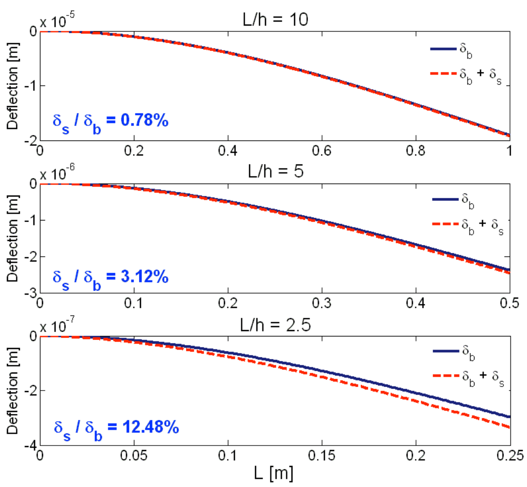


Figure 10. Mode shapes of elastic cantilever beam with varying *L/h*.

CONCLUSION

The selection criteria for Euler-Bernoulli or Timoshenko beam theories are generally given by means of some deterministic rule involving beam dimensions. The Euler-Bernoulli beam theory is used to model the behaviour of flexure-dominate beams (long beams) and the Timoshenko beam theory applies for shear-dominated beams (short beams). In between, many other theories should be equivalent, and some agreement between them would be expected. In this paper, it is shown that, for some mid-length beams, analytical natural frequencies for the eight beam theories (Euler-Bernoulli, Rayleigh, Euler-Bernoulli modified, Bress, Volterra, Ambartsumyan, Timoshenko and Love) agree very well under high natural frequencies. In this case, all the theories are equivalent when compared analytically.

Numerical method as finite element model is very useful but stiffer than the real structure. In general, displacement results are smaller in magnitudes than the exact values. Furthermore, the finite element solution depends strongly on the number of element. Hence, finite element solution of natural frequency and mode shapes provide a lower bound of the analytical solution.

REFERENCES

1. Hjelmstad, K.D. *Fundamentals of Structural Mechanics.* 2nd Ed. Springer, Berlin, 2005.
2. Thorby, D., *Structural Dynamics and Vibration in Practice: An Engineering Handbook.* Butterworth-Heinemann, Oxford, 2008.
3. Yang, B. *Stress, Strain, and Structural Dynamics.* Elsevier Academic Press, Burlington, 2005.
4. Gasch, R., Knothe, K., Liebich, R. *Strukturdynamik: Diskrete Systeme und Kontinua.* 2.Aufl. Springer-Vieweg, Berlin, 2012.
5. Freymann, R. *Strukturdynamik: Ein anwendungsorientiertes Lehrbuch.* Springer, Berlin, 2011.
6. Schumacher, A. *Optimierung mechanischer Strukturen: Grundlagen und industrielle Anwendungen.* 2.Aufl. Springer-Vieweg, Berlin, 2013.
7. Beards, C.F. *Structural Vibration: Analysis and Damping.* Butterworth-Heinemann, Oxford, 2003.

8. Karnovsky, I.A., Lebed, O.I. *Formula for Structural Dynamics.* McGraw-Hill, New York, 2004.
9. Liong, R.T. and Proppe, C. *Maschinendynamik mit* MATLAB®, Script der Vorlesung der Maschinendynamik, Institut für Technische Mechanik, KIT, Karlsruhe, Germany, 2012.